

# Killing Correspondence in Finsler spaces

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## Abstract

The present paper deals with the Killing correspondence between some Finsler spaces. We consider a Finsler space equipped with a  $\beta$ -change of metric and study the Killing correspondence between the original Finsler space and the Finsler space equipped with  $\beta$ -change of metric. We obtain necessary and sufficient condition for a vector field Killing in the original Finsler space to be Killing in the Finsler space equipped with  $\beta$ -change of metric. Certain consequences of such result are also discussed.

**Keywords:** Finsler spaces,  $\beta$ -change, Killing vector field

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## 1 Introduction

As a matter of investigation, it is important to observe how properties of a Finsler space change under a change in the metric. Several geometers from different parts of the globe have been working in this direction for the last 2-3 decades. M.S. Knebelman [1], S.Golab [2] and M.Hashiguchi [3] studied conformal change of Finsler metrics. Park and Lee [4] discussed various Randers changes of Finsler spaces with  $(\alpha, \beta)$ -metrics of Douglas type. M. Matsumoto [6] and T. Aikou [5] studied and investigated several properties of projective change and projective Randers change. In 1984, C. Shibata [8] studied  $\beta$ -change of Finsler metrics and discussed certain invariant tensors under such a change.

Killing equations play important role in the study of a Finsler space which undergoes a change in the metric. In fact, they give an equivalent characterization for the transformations to preserve distances. In 1979, Singh, *et. al.* [7] studied a Randers space  $F^n \left( M, L(x, y) = (g_{ij}(x) y^i y^j)^{\frac{1}{2}} + b_i(x) y^i \right)$ ,  $n \geq 2$  which undergoes a change  $L(x, y) \mapsto L^*(x, y) = L^2(x, y) + (\alpha_i(x) y^i)^2$ . They discussed Killing correspondence of the spaces  $F^n(M, L)$  and  $F^{*n}(M, L^*)$ .

In the present paper, we consider a general Finsler space  $F^n(M, L)$  which undergoes a  $\beta$ -change, that is  $L(x, y) \mapsto \bar{L}(x, y) = f(L, \beta)$ , where  $\beta(x, y) = b_i(x) y^i$  is a 1-form. We study Killing correspondence of the Finsler spaces  $F^n(M, L)$  and  $\bar{F}^n(M, \bar{L})$ . For the notations and terminology, we refer the reader to the books [9] and [10], and the paper [8] by Shibata.

The paper is organized as follows. In section 2, we give some preliminaries which are used in the discussion of subsequent sections. Section 3 deals with Killing correspondence of  $F^n(M, L)$  and  $\bar{F}^n(M, \bar{L})$ , where  $\bar{L}(x, y) = f(L, \beta)$ . In section 4, we give conclusion to the results obtained in the paper and discuss future possible work to be done in this direction.

## 2 Preliminaries

Let  $F^n(M, L)$ ,  $n \geq 2$  be an  $n$ -dimensional Finsler space. Suppose that the metric function  $L(x, y)$  undergoes a change  $L(x, y) \mapsto \bar{L}(x, y) = f(L, \beta)$ , where  $\beta(x, y) = b_i(x) y^i$  is a 1-form and the new space is  $\bar{F}^n(M, \bar{L})$ . This change of metric is called a  $\beta$ -change (*see* [8] and [9]).

The angular metric tensor  $\bar{h}_{ij}$  of the space  $\bar{F}^n$  is given by [8]

$$(2.1) \quad \bar{h}_{ij} = p h_{ij} + q_0 m_i m_j,$$

where

$$(2.2) \quad \begin{cases} p = f f_1 / L, & q_0 = f f_2, & m_i = b_i - \beta y^i / L^2, \\ f_1 = \partial f / \partial L, & f_2 = \partial f / \partial \beta, \end{cases}$$

$h_{ij}$  being the angular metric tensor of  $F^n$ . The fundamental metric tensor  $\bar{g}_{ij}$  and its inverse  $\bar{g}^{ij}$  of  $\bar{F}^n$  are expressed as [8]

$$(2.3) \quad \bar{g}_{ij} = pg_{ij} + p_0 b_i b_j + p_{-1}(b_i y_j + b_j y_i) + p_{-2} y_i y_j,$$

$$(2.4) \quad \bar{g}^{ij} = g^{ij}/p - sb^i b^j - s_{-1}(b^i y^j + b^j y^i) - s_{-2} y^i y^j,$$

where

$$(2.5) \quad \left\{ \begin{array}{l} p_0 = q_0 + f_2^2, \\ q_{-1} = f f_{12}/L, \quad p_{-1} = q_{-1} + p f_2/f, \quad q_{-2} = f(f_{11} - f_1/L)/L^2, \\ p_{-2} = q_{-2} + p^2/f^2, \\ b^i = g^{ij} b_j, \quad b^2 = g^{ij} b_i b_j, \quad s_0 = \bar{L} q_0/(\tau p L^2), \\ s_1 = p_{-1} \bar{L}^2/(\tau p L^2), \quad s_{-2} = p_{-1}(\nu p L^2 - b^2 \bar{L}^2)/(\tau p L^2 \beta), \\ \tau = \bar{L}^2(p + \nu q_0)/L^2, \quad \nu = b^2 - \beta^2/L^2, \end{array} \right.$$

$g_{ij}$  and  $g^{ij}$  respectively being the metric tensor and inverse metric tensor of  $F^n$ . The Cartan tensor  $\bar{C}_{ijk}$  and the associate Cartan tensor  $\bar{C}_{ij}^h$  of  $\bar{F}^n$  are given by the following expressions:

$$(2.6) \quad \bar{C}_{ijk} = p C_{ijk} + \frac{1}{2} p_{-1} \mathfrak{S}_{(ijk)} \{h_{ij} m_k\} + \frac{1}{2} p_{02} m_i m_j m_k,$$

$$(2.7) \quad \bar{C}_{ij}^h = C_{ij}^h - V_{ij}^h,$$

where

$$(2.8) \quad \begin{aligned} V_{ij}^h = & Q^h (p C_{imj} b^m - p_{-1} m_i m_j) - \left( \frac{1}{p} m^h - \nu Q^h \right) (p_{02} m_i m_j + p_{-2} h_{ij})/2 \\ & - p_{-1} (h_i^h m_j + h_j^h m_i)/(2p), \end{aligned}$$

$$(2.9) \quad Q^h = s_0 b^h + s_{-1} y^h, \quad h_i^h = g^{hr} h_{ir}, \quad m^h = g^{hr} m_r, \quad p_{02} = \partial p_0 / \partial \beta,$$

$\mathfrak{S}_{(ijk)}$  denote the cyclic sum with respect to the indices  $i, j$  and  $k$ ;  $C_{ijk}$  and  $C_{ij}^h$  respectively being the Cartan tensor and associate Cartan tensor of  $F^n$ .

The spray coefficients  $\bar{G}^i$  of  $\bar{F}^n$  in terms of the spray coefficients  $G^i$  of  $F^n$  are expressed as [8]

$$(2.10) \quad \bar{G}^i = G^i + D^i,$$

where

$$\begin{aligned} D^i &= (q/p) F_0^i + (p E_{00} - 2q F_{r0} b^r)(s_{-1} y^i + s_0 b^i)/2, \\ F_j^i &= g^{ir} F_{rj}, \quad E_{jk} = (1/2)(b_{j|k} + b_{k|j}), \quad F_{jk} = (1/2)(b_{j|k} - b_{k|j}), \end{aligned}$$

the symbol ' $|$ ' denote the  $h$ -covariant derivative with respect to the Cartan connection  $CT$  and the lower index ' $_0$ ' (except in  $s_0$ ) denote the contraction by  $y^i$ .

The relation between the coefficients  $\bar{N}_j^i$  of Cartan nonlinear connection in  $\bar{F}^n$  and the coefficients  $N_j^i$  of the corresponding Cartan nonlinear connection in  $F^n$  is given by [8]

$$(2.11) \quad \bar{N}_j^i = N_j^i + D_j^i,$$

where

$$(2.12) \quad D_j^i = \dot{\partial}_j D^i, \quad \dot{\partial}_j \equiv \partial/\partial y^j.$$

The coefficients  $\bar{F}_{jk}^i$  of Cartan connection  $C\bar{\Gamma}$  in  $\bar{F}^n$  and the coefficients  $F_{jk}^i$  of the corresponding Cartan connection  $CT$  in  $F^n$  are related as [8]

$$(2.13) \quad \bar{F}_{jk}^i = F_{jk}^i + D_{jk}^i,$$

where

$$\begin{aligned} D_{jk}^i &= \{(1/p)g^{is} - Q^i b^s - y^s(s_{-1} b^i + s_{-2} y^i)\} \\ &\quad (B_{sj} b_{0|k} + B_{sk} b_{0|j} - B_{kj} b_{0|s} + F_{sj} Q_k + F_{sk} Q_j + E_{kj} Q_s + p C_{jkr} D_s^r \\ &\quad + V_{jkr} D_s^r - p C_{skm} D_j^m - V_{sjm} D_k^m - p C_{sjm} D_k^m - V_{skm} D_j^m); \end{aligned}$$

$$B_{kj} = 2\dot{\partial}_j Q_k.$$

The difference tensor  $D_{jk}^i$  satisfies the following properties:

$$(i) \quad D_{j0}^i = B_{j0}^i = D_j^i, \quad (ii) \quad D_{00}^i = 2D^i, \quad \text{where} \quad B_{jk}^i = \dot{\partial}_k D_j^i.$$

### 3 Killing Correspondence of $F^n$ and $\bar{F}^n$

Let us consider an infinitesimal transformation

$$(3.1) \quad 'x^i = x^i + \epsilon v^i(x),$$

where  $\epsilon$  is an infinitesimal constant and  $v^i(x)$  is a contravariant vector field.

The vector field  $v^i(x)$  is said to be a Killing vector field in  $F^n$  if the metric tensor of the Finsler space with respect to the infinitesimal transformation (3.1) is Lie invariant, that is

$$(3.2) \quad \mathfrak{L}_v g_{ij} = 0,$$

$\mathfrak{L}_v$  being the operator of Lie differentiation. Equivalently, the vector field  $v^i(x)$  is Killing in  $F^n$  if

$$(3.3) \quad v_{i|j} + v_{j|i} + 2C_{ij}^h v_{h|0} = 0,$$

where  $v_i = g_{il} v^l$ .

Now, we prove the following result which gives a necessary and sufficient condition for a Killing vector field in  $F^n$  to be Killing in  $\bar{F}^n$ :

**Theorem 3.1.** *A Killing vector field  $v^i(x)$  in  $F^n$  is Killing in  $\bar{F}^n$  if and only if*

$$(3.4) \quad V_{ij}^h v_{h|0} + C_{rjl} v^l D_i^r + C_{ril} v^l D_j^r + v_r D_{ij}^r + \bar{C}_{ij}^h (2C_{rhl} v^l D^r + v_r D_h^r) = 0,$$

where  $\bar{C}_{ij}^h$  is the associate Cartan tensor of  $\bar{F}^n$ .

*Proof.* Assume that  $v^i(x)$  is Killing in  $F^n$ . Then (3.3) is satisfied. By definition, the  $h$ -covariant derivatives of  $v_i$  with respect to  $C\bar{\Gamma}$  and  $C\Gamma$  are respectively given as

$$(3.5) \quad (a) \quad v_{i||j} = \partial_j v_i - (\dot{\partial}_r v_i) \bar{G}_j^r - v_r \bar{F}_{ij}^r, \quad (b) \quad v_{i|j} = \partial_j v_i - (\dot{\partial}_r v_i) G_j^r - v_r F_{ij}^r,$$

where  $\partial_j \equiv \partial/\partial x^j$  and  $'_{||}'$  denote the  $h$ -covariant differentiation with respect to  $C\bar{\Gamma}$ . Equation (3.5)(a), by virtue of (2.10), (2.13) and (3.5)(b), takes the form

$$(3.6) \quad v_{i||j} = v_{i|j} - 2C_{ril} v^l D_j^r - v_r D_{ij}^r.$$

Now, from (3.6), we have

$$(3.7) \quad \begin{aligned} v_{i||j} + v_{j||i} + 2\bar{C}_{ij}^h v_{h||0} = & v_{i|j} + v_{j|i} + 2\bar{C}_{ij}^h v_{h|0} - 2C_{ril} v^l D_j^r - 2C_{rjl} v^l D_i^r \\ & - 2v_r D_{ij}^r - 2\bar{C}_{ij}^h (2C_{rhl} v^l D^r + v_r D_h^r). \end{aligned}$$

Using (2.7) in (3.7) and applying (3.3), we get

$$(3.8) \quad \begin{aligned} v_{i||j} + v_{j||i} + 2\bar{C}_{ij}^h v_{h||0} = & -2V_{ij}^h v_{h|0} - 2C_{ril} v^l D_j^r - 2C_{rjl} v^l D_i^r \\ & - 2v_r D_{ij}^r - 2\bar{C}_{ij}^h (2C_{rhl} v^l D^r + v_r D_h^r). \end{aligned}$$

Proof completes with the observation that  $v^i(x)$  is Killing in  $\bar{F}^n$  if and only if  $v_{i||j} + v_{j||i} + 2\bar{C}_{ij}^h v_{h||0} = 0$ , that is, if and only if (3.4) holds.  $\square$

If a vector field  $v^i(x)$  is Killing in  $F^n$  and  $\bar{F}^n$ , then from Theorem 3.1, (3.4) holds, which on transvection by  $y^i$  yields

$$(3.9) \quad 2C_{rjl} v^l D^r + v_r D_j^r = 0.$$

Equation (3.4), in view of (3.9), enables us to state the following:

**Corollary 3.1.** *If a vector field  $v^i(x)$  is Killing in  $F^n$  and  $\bar{F}^n$ , then*

$$(3.10) \quad V_{ij}^h v_{h|0} + C_{rjl} v^l D_i^r + C_{ril} v^l D_j^r + v_r D_{ij}^r = 0.$$

As another important consequence of Theorem 3.1, we have the following:

**Corollary 3.2.** *If a vector field  $v^i(x)$  is Killing in  $F^n$  and  $\bar{F}^n$ , then the vector  $v_i(x, y)$  is orthogonal to the vector  $D^i(x, y)$ .*

*Proof.* As  $v^i(x)$  is Killing in  $F^n$  and  $\bar{F}^n$ , (3.4) holds, which on transvection by  $y^i$  gives (3.9). Again transvecting (3.9) by  $y^j$ , it follows that  $v_r D^r = 0$ . This proves the result.  $\square$

## 4 Discussion and Conclusion

We proved Theorem 3.1 as the main result and as its consequences we obtained Corollary 3.1 and Corollary 3.2. Since the Killing equation (3.2) is a necessary and

sufficient condition for the transformation (3.1) to be a motion in  $F^n$  (*vide* [10]), the condition (3.4) obtained in Theorem 3.1 may be taken as the necessary and sufficient condition for the vector field  $v^i(x)$ , generating a motion in  $F^n$ , to generate a motion in  $\bar{F}^n$  as well. Also, since every motion is an affine motion and every affine motion is a projective motion (*vide* [11]-[13]), it is clear that vector field  $v^i(x)$ , generating an affine motion (respectively projective motion) in  $F^n$ , generates an affine motion (respectively projective motion) in  $\bar{F}^n$  if condition (3.4) holds. The main result and its consequences, obtained in the paper, may be further utilized to link various transformations in  $F^n$  with corresponding transformations in  $\bar{F}^n$ .

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